**MATH 158:**

**ALGEBRA**

Semester 2

Lecture Notes

*Compiled By*

SAMPSON TWUMASI-ANKRAH (Ph.D)

DEPARTMENT OF MATHEMATICS, KNUST

Contents

[**DIFFERENTIATION** 3](#_Toc505755342)

[SLOPE / GRADIENT OF A LINE 3](#_Toc505755343)

[DIFFERENTIATION OF SUM AND DIFFERENCES 4](#_Toc505755344)

[DIFFERENTIATION OF A CONSTANT 5](#_Toc505755345)

[SPECIAL CASE 5](#_Toc505755346)

[DIFFERENTIATING LOGARITHM FUNCTIONS 6](#_Toc505755347)

[DIFFERENTIATING OF EXPONENTIAL FUNCTIONS 6](#_Toc505755348)

[PRODUCT RULE 7](#_Toc505755349)

[PRODUCT OF A QUOTIENT 8](#_Toc505755350)

[DIFFERENTIATION OF A FUNCTION OF A FUNCTION (CHAIN RULE) 8](#_Toc505755351)

[IMPLICIT DIFFENTIATION 9](#_Toc505755352)

[Differentiation of Trigonometric Functions 10](#_Toc505755353)

[Maxima and Minima 10](#_Toc505755354)

[Tests for Points 11](#_Toc505755355)

[Applications 14](#_Toc505755356)

[**LIMITS** 19](#_Toc505755357)

[Theorem: Uniqueness of the limit 19](#_Toc505755358)

[Operations with Limits 19](#_Toc505755359)

[One – Sided Limits 22](#_Toc505755360)

[Limit as  and as  23](#_Toc505755361)

[L’ HOSPITAL’S RULE 24](#_Toc505755362)

UNIT 1

# DIFFERENTIATION

Differentiation is the process of finding the derivative of a function. The derivative of a function is also called its derived function and also its derived coefficient.

The derivative of *y* with respect to *x* is usually written as 

|  |  |
| --- | --- |
| FUNCTION | DERIVATIVE |
| Constant | 0 |
|  |  |
|  |  |
|  |  |

## SLOPE / GRADIENT OF A LINE

A (3, 2) and B (5, 8)

The gradient is given by the formula,

is the rate of change in with respect to .

For example;

## DIFFERENTIATION OF SUM AND DIFFERENCES

Here we differentiate term by term. For example;

1. If + , find
2. or +

## DIFFERENTIATION OF A CONSTANT

If , find

can be written as

Whenever we differentiate a constant the result is 0. That is,

If find

If , find

## SPECIAL CASE

If , then

If , then

If ,

## DIFFERENTIATING LOGARITHM FUNCTIONS

If 

1. , find
2. , find
3. , find

## DIFFERENTIATING OF EXPONENTIAL FUNCTIONS

If 

**Example**

1. , find

Differentiate the power and multiply it by the same function.

1. If

1. If

## PRODUCT RULE

If and are both function of , and , then

Examples:

1. If find

1. If

## PRODUCT OF A QUOTIENT

If and are both function of , and ,

Example: find

## DIFFERENTIATION OF A FUNCTION OF A FUNCTION (CHAIN RULE)

Example:

1. find

1. If find

1. If find

1. Given that find

## IMPLICIT DIFFENTIATION

Examples:

1. If , find

Differentiate with respect to

1. If , find

Differentiate with respect to

Find the gradient of the curve at .

1. If find

Differentiate with respect to

1. If find

Differentiate with respect to

## Differentiation of Trigonometric Functions

## Maxima and Minima

At a point of local maximum, a function has a greater Value than at points immediately on either side of it. At a point of local minimum, a function has a smaller Value than at points immediately on either side of it. Local maxima and minima are also called turning points.

A function may have more than one turning point. The local maxima and minima are not necessarily the greater or least Values of the function in the given range.

maximum

maximum

minimum

greatest

minimum

least

**Points of Inflexion**

At a point of inflexion, the graph of the function changes the direction in which it is curving.











For general points inflection 

### Tests for Points

A stationary point is a point at which  . Local maxima, minima and horizontal points of inflexion are stationary points. To test for stationary points,

a) Find and 

b) Put  and solve the resulting equation to find the x – coordinate(s) of the point(s)

c) Find  at the stationary point(s).

i.) if, the point is local maximum

ii.) if, the point is local minimum

iii.) if, find the sign of  for a value of *x* just to the left and just to the right of the point.

|  |  |  |
| --- | --- | --- |
| Sign to the Left | Sign to the Right | Types of point |
| + | - | Maximum |
| - | + | Minimum |
| +  - | +  - | point of inflexion |

To test general points of inflexion .

* 1. Find v
  2. Put  and solve the resulting equation to find the possible – coordinate(s)

c) Find the sign of  for a value of  just to the left and to the right of the point. If  changes sign, the point is a point of inflexion.

**Example**

Find the stationary points of  and identify their nature.

**Solution**







At stationary points, i.e.,  , 



When x = 3, 



Therefore (3, 0 ) is a local minimum.

When 



Therefore (1,  ) is a local maximum.

**Example**

Find any points of inflexion of.

**Solution**







At a general point of inflexion, i.e., 2*x* – 4 = 0 ⇒ *x* = 2

For  = 2+, i.e.  changes sign

For *x* = 2-, 

So  is a general point of inflexion.

## Applications

Maxima and Minima can be applied to practical problems in which the maximum or minimum value of a quantity is required. The procedure is

a) Write an expression for the required quantity.

b) Use the given conditions to rewrite it in terms of a single variable.

c) Find the turning point(s) and their type(s). It is often obvious from the problem itself whether a maximum or minimum has been obtained.

**Examples**

1. A rectangle has perimeter 28m. What is its maximum area?

**Solution**

Let  and  meters be the sides of the rectangle.

Its perimeter = 



It’s Area, 



When A is a maximum, = 0 i.e., 14 –



Since 



 (max)

Therefore the maximum area is 72m2 = 49m2.

2. The sides of a rectangular sheet of metal are 8cm and 3cm. A square of side *x*cm is cut from each corner of the sheet and the remaining piece is folded to make an open box.

a) Show that the volume *V* of the box is given by cm3

b) Find the value of  for which the volume of the box is a maximum. Calculate the maximum volume.

**Solution** *x*cm

3cm

8cm

The volume of the box 

cm

b) .

For maximum (or minimum) value of ,  = 0

I.e., 







Clearly  cannot be 3cm since the width of the sheet initially is only 3cm. So x = 2/3cm

 = ,

When,  = 24() – 44 = 16 – 44 = – 28 < 0

: . *V* is a maximum.

So the maximum volume is given by substituting  in (1) giving Vmax =  = 711/27 cm3

3. A manufacture of tennis balls produces  hundred balls per week. The cost of production is

 and the revenue returned is .

How many balls should be made each week for maximum profit?

**Solution**

The profit function *P(x)* in producing *x* hundred balls is 

i.e., Profit = revenue – cost.





Now for maximum profit, = 0

 = 



= -0.2 < 0 (max)

The manufacture should make 25 hundred balls per week

4. Find two positive numbers whose product is as large as possible.

**Solution**

Let one of the numbers be and the other. Since the sum is 18, it follows that 

Now the product  is a function of both and.

 but 





For a maximum, we require = 0

 = = 0

So  and hence, .

= -2 < 0 (max)

Hence, the numbers are 9 and 9.

UNIT 2

# LIMITS

## Theorem: Uniqueness of the limit

If  and  then .

In the case of a standard polynomial, the value given by the limit and the value of the polynomial are the same.

**Example**

In cases where evaluating a fractional expression at a point would lead to undefined value of , one first tries to factor the expression and divide out any common term before evaluating.

**Example**



### Operations with Limits

**THEOREM**

Let  and  be functions such that  and 

Then

a)  .

b)  , for any constant :

c) 

d) , provided ≠0

e) 

**Examples**

1. Find 

**Solution**

As 



2. Find 

**Solution**

=

3. Find 

**Solution**

4. Find 

**Solution**

= 

the function has no limit.

5. Find 

**Solution**

==3

6. Find 

**Solution**



Now 

: . 

7. Find 

**Solution**



### One – Sided Limits

It is also possible to compute a one sided limit of an expression. In these, we determine whether one approaches the limit point from positive side (called a right-hand limit and denoted )

I.e. The number  is called the right-hand limit of *f* at *C* if  approaches  as *x* approaches subject to the requirement that .

Or from the negative side (called the left–hand limit and denoted  I.e., the left–hand limit is similar to the right – hand limit except only points are considered.

**Example**

Consider the function 

In this case,  and furthermore, .

As usual, these results do not depend on the value of, or even on whether or not f is defined at .

**Theorem**

The function *f* has the limit *L* at the point *C* if and only if the left–hand limit and the right–hand limit at exist and are both equal to. That is  if and only if .

Therefore in this example, 

Find: a) 

b) 

**Solution**

a)  = 

The limit of the numerator is 14. To find . Thus, the limit of the denominator is 0, as the denominator is approaching 0 through positive values. Consequently, .

b) 

In this case, the limit of the denominator is again zero, since the denominator is approaching zero through negative values.

I.e., for the denominator we have, 

### Limit as and as

If  approaches a number  as  →∞, then we write .

In similar way, we write if  approaches a number  as →-∞.

**Examples**

1. Let .

As  or as,  takes on values closer and closer to zero.

Thus  =.

## L’ HOSPITAL’S RULE

Suppose we have to find the limiting value of a function  at, when direct substituting of *g(x,)*  gives the indeterminate form  i.e., at ,  and .

In this case, the ratio of the differential coefficients of the numerator and denominator at *x* = *c* provided, of course, that both  and  are not zero themselves.

: .

This is known as L’ HOSPITAL’S rule and is extremely useful for finding limiting values when the differential coefficients of the numerator and denominator can easily be found.

**Example**

Find 

**Solution**

= (Indeterminate)

=1

: . =1

**Example**

Find  

**Solution**

 Direct substitution gives, so we apply L’ HOSPITAL’S rule which gives 

Note

L’ HOSPITAL’S rule applies only when indeterminate form arises.

**Example**

Determine 

**Solution**

 ( indeterminate )

Applying L’HOSPITAL’S rule, we have

